

Spatio-temporal structure of traffic flow in a system with an open boundary

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The spatio-temporal structure of traffic flow pattern is investigated under the open boundary condition using the optimal velocity (OV) model. The parameter region where the uniform solution is convectively unstable is determined. It is found that a localized perturbation triggers a linearly unstable oscillatory solution out of the linearly unstable uniform state, and it is shown that the oscillatory solution is also convectively stabilized. It is demonstrated that the observed traffic pattern near an on-ramp can be interpreted as the noise sustained structure in the open flow system.

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The complex behavior of traffic flow on a freeway has been attracting the research interests [1,2]. In recent years, various kind of dynamical states have been observed in real traffic; a peculiar fluctuating flow called *the synchronized flow* (SF) was found in the high density region which could result in the jammed flow [1,2]. The flux of cars is higher than the jammed flow, but fluctuates synchronously between different lanes. It has been found that SF is triggered by a localized perturbation such as an on-ramp, and may persist for several hours. *The stop-and-go state* (SGS) is another state of traffic where the traffic goes through an alternating pattern of jammed and free flows in a short period of space and time; SGS is often observed behind SF region [2]. The hysteresis has been also found in the transition between the jammed flow or SF and the free flow when the car density changes [1,2].

One of the major topics in the current theoretical research on the traffic flow is to make clear the characteristics of the complex behavior [3–6]. Recently, the effects of the localized perturbation were investigated in the hydrodynamical traffic flow models with an on-ramp under the open boundary condition [3]. It has been found that the influx perturbation at the on-ramp causes various types of oscillatory flows as well as the convectively stabilized uniform dense flow, but properties of these oscillatory flows and spatio-temporal patterns have not been understood yet.

The optimal velocity (OV) model is another type of model based on the driving behavior of individual cars [7]. The OV model has been demonstrated to show the transition from free flow to jammed flow, and to reproduce the flux-density diagram (the fundamental diagram) similar to the one observed in the real traffic [8]. The authors have examined the effects of a localized perturbation in the OV model and found that the localized perturbation triggers the oscillatory flow out of the linearly unstable uniform flow [4].

In the present work, we study the convective instability in the OV model and clarify the mechanism of the spatio-temporal pattern triggered by a localized perturbation. We show the resulting spatio-temporal pattern is analogous to the one observed in the real traffic and can be understood using the idea of the noise sustained structure in the open flow system [9].

The OV model [7] is the one-dimensional car-following model where each driver tends to drive at the optimal velocity determined by the headway of his car. The position of the n th car $x_n(t)$ at time t obeys the equation of motion

$$\ddot{x}_n(t) = a [U(b_n(t)) - \dot{x}_n(t)]; \quad b_n(t) \equiv x_{n+1} - x_n, \quad (1)$$

where the dots denote the time derivative and b_n represents the headway of the n th car; we assume the $(n+1)$ th car precedes the n th car. The parameter a is a sensitivity constant, and the function $U(b)$ determines the optimal velocity for a driver when his headway is b . For $U(b)$, we employ

$$U(b) = \tanh(b - 2) + \tanh(2), \quad (2)$$

as other works on the OV model [7,10].

Equation (1) has a uniform solution

$$x_n(t) = \bar{b}n + U(\bar{b})t, \quad (3)$$

where all the cars go with the same headway \bar{b} and the same speed $U(\bar{b})$. The dispersion for the small deviation around the uniform solution in the “index frame” is given by

$$\omega = -i\frac{a}{2} + \frac{i}{2}\sqrt{a^2 + 4aU'(\bar{b})(e^{ik} - 1)} \equiv \omega_I(k) \quad (4)$$

where ω and k denote the angular frequency and the wave number in the index frame; $\exp(ikn - i\omega t)$. From this, it can be seen that the uniform solution is linearly unstable when

$$a < 2U'(\bar{b}), \quad (5)$$

where the prime denotes the derivative by its argument [7]. When we perturb the linearly unstable uniform solution under the periodic boundary condition, the effect of perturbation grows in time and eventually the system segregates into two regions; the jammed region with smaller headway and lower velocity, and the free flow region with larger headway and higher velocity [7]. It is also known that Eq. (1) is reduced to the Korteweg-de Vries (KdV) or the modified KdV equation in weak nonlinear analysis near the linear stability limit [10].

We now study the system behavior in the situation where the upper and lower stream are distinguished, which is more appropriate to the actual traffic. We employ the open boundary condition defined as follows: (i) At the upper stream end ($x = 0$), cars with the velocity $U(\bar{b})$ enter the system with the constant time interval $\bar{b}/U(\bar{b})$. (ii) Around the lower stream end, the car that is farthest ahead, which has no car to follow within the system, obeys the equation of motion $\ddot{x}_f = a[U(\bar{b}) - \dot{x}_f]$, and it goes out of the system at $x = L$. Here, \bar{b} is chosen to fit the uniform initial state, which will go on if there is no perturbation.

The uniform solution is perturbed locally in space and time by shifting the velocity of the 0th car at $t = 0$ by a small value ϵ . In actual simulations, the initial condition is given as:

$$x_n(0) = \bar{b}n + \frac{L}{2}, \quad \dot{x}_n(0) = U(\bar{b}) \quad \text{for } n = \pm 1, \pm 2, \dots, \quad (6)$$

$$x_0(0) = \frac{L}{2}, \quad \dot{x}_0(0) = U(\bar{b}) + \epsilon. \quad (7)$$

Within the parameter region where the initial uniform solution is linearly unstable, there are following two regions: When a is larger than a critical value a_c which depends on \bar{b} ($a_c = a_c(\bar{b}) < a < 2U'(\bar{b})$), the disturbance travels only upstream (Fig. 1(a)). Therefore, the disturbed region eventually goes out from the system, and the linearly unstable uniform solution is recovered. On the other hand, when $a < a_c(\bar{b})$, the disturbance travels in both directions (Fig. 1(b)), and the uniform flow region is eliminated completely. In the former case, the uniform solution is *convectively unstable*; the growing perturbation is convected away from any fixed location. The uniform solution is *absolutely unstable* in the latter case; the perturbation grows at every point in space [9,11]. The boundary between the convective instability and the absolute one depends on the reference frame. It should be noted that, for a car-following model where one does not feel any effects from behind as in the OV model, the instability of the linearly unstable uniform solution is always convective in the index frame, that moves with the cars.

The stability in the *laboratory frame* can be examined by following the procedure described in Ref. [11]. The dispersion relation in the laboratory frame is given by

$$\omega(k) = k \frac{U(\bar{b})}{\bar{b}} + \omega_I(k) \quad (8)$$

because the laboratory frame is moving with the speed $-U(\bar{b})/\bar{b}$ relative to the index frame. The convective stability limit $a_c(\bar{b})$ is determined for a given \bar{b} by the set of equations;

$$\left. \frac{d\omega(k)}{dk} \right|_{k=k_c} = 0, \quad \Im[\omega(k_c)] = 0. \quad (9)$$

Fig. 2 shows numerical estimate of Eq. (9) for the boundary $a = a_c(\bar{b})$ (the solid line) with the linear stability limit $a = 2U'(\bar{b})$ (the dashed line). The uniform solution is convectively unstable in the region between the solid line and the dashed line ($a_c(\bar{b}) < a < 2U'(\bar{b})$).

Another characteristic of the density diagram is its spatio-temporal pattern; the oscillatory flow (a regular stripe in Fig. 1(b)) followed by an alternating sequence of jams and free flows (an irregular stripe with stronger contrast in Fig. 1(b)). This structure is triggered out of the linearly unstable uniform solution by the localized perturbation.

This sequence can be seen more clearly in Fig. 3, in which the time evolution of the headway of the -578th car is shown. Each car travels through the sequence, thus the time evolution of the headway shows the structure of flow from the upper to the lower stream. In Fig. 3(a), the car passes through the alternating jammed and free flow region during $1000 \lesssim t \lesssim 1400$, and the oscillatory flow region during $1600 \lesssim t \lesssim 1800$. We can see the periodic behavior of the headway in the oscillatory flow region (Fig. 3(c)).

This oscillatory behavior of the solution can be expressed by

$$b_n = \hat{b} + f(n - ct) \quad (10)$$

with an appropriate phase speed c and a function f , which, we assume, has zero mean by taking \hat{b} as the mean headway. We have already found that Eq. (10) can be a solution of the original equation of motion (1) for a finite range of the phase speed c [4]; e.g., $c \in (-0.637, -0.556)$ for $a = 1.0$ and $\hat{b} = 2.0$. The shape, the wavelength, and the amplitude of the solution (10) depend upon the parameters a , \hat{b} , and c . When we set the phase speed at c_s , the value obtained from the direct simulation of Eq. (1), namely, $c = c_s = -0.610$ for $a = 1.0$ and $\hat{b} = 2.0$, we get $f(n - c_s t)$ which coincides with the results of the simulation (see Fig. 5(c) in Ref. [4]).

The linear stability of periodic solutions can be determined by the Floquet exponent, or a complex linear growth rate averaged over the period [12]. We calculate the Floquet exponents of the oscillatory solution for the N-car system with the periodic boundary condition, where the headway of the $N + 1$ th car equals to that of the first car, and the fixed boundary condition, where the headway of the $N + 1$ th car obeys the oscillatory solution. It is found that the maximum value of the real part of the exponents is always positive under the periodic boundary condition, while it is always negative under the fixed boundary condition. This implies the oscillatory solution is linearly unstable, but the growing disturbance is convected away if the headway of the foremost car obeys the oscillatory solution; namely, the oscillatory solution is convectively unstable in the index frame.

We now analyze the mechanism how the solution with a particular phase speed is chosen out of the finite range of allowed ones. Suppose the downstream front of the oscillatory region is propagating with the speed V_0 relative to the index frame, then V_0 and the wave number k_f in the index frame can be determined by the condition that the linear growth rate in the frame moving with the front is zero;

$$\left. \frac{d\omega_{V_0}(k)}{dk} \right|_{k=k_f} = 0, \quad \Im[\omega_{V_0}(k_f)] = 0, \quad (11)$$

where $\omega_{V_0}(k) \equiv \omega_I(k) - kV_0$ is the dispersion in the moving frame. The angular frequency of the oscillation at the front in the moving frame is given by $\Re[\omega_{V_0}(k_f)]$, and this oscillation is amplified as it travels upstream to induce the oscillatory solution. It is natural to expect that the time period of the oscillatory solution is the same as that of the downstream oscillation in the moving frame because each car tends to follow the motion of the preceding car (see Eq. (1)). Therefore, there should be the relationship between the wave length λ of the oscillatory solution and the phase speed c_s ,

$$\lambda = (|c_s| + V_0) \frac{2\pi}{|\Re[\omega_{V_0}(k_f)]|}. \quad (12)$$

This was confirmed by the simulations as is shown in Table I.

This oscillatory flow cannot extend over the whole system because it is only convectively stable. The motions of cars in the upper stream gradually deviate from the oscillatory solution, and eventually the oscillatory flow breaks up (Fig. 3(d)). As a result, the alternating sequence of jams and free flows is formed behind the oscillatory flow region. The alternating region cannot be completely periodic because any infinitesimal perturbation grows as it travels upstream (Fig. 3(e)). This mechanism of the structure formation is general to convectively unstable open flow systems, such as the complex Ginzburg-Landau equation with an advection term [9].

Before concluding, we discuss the present results in connection with the other traffic models and the observation of real traffic. As we have seen, the convective instability in the index frame plays important role in the structure formation. It should be a common feature of car-following models and the main feature of the flow pattern obtained should not depend on the details of the models [14].

The hydrodynamical models based on the continuum description have been found to show the similar behavior to the oscillatory flow [3]. The oscillatory flows appear near an on-ramp with influx in the simulations under the open boundary condition [15]. It has been also found that the convectively unstable uniform flow can be realized in the upper stream of the on-ramp in some parameter region. Therefore, it is natural to expect that a *noise-sustained structure* [9] similar to the one discussed in the present work appears in the hydrodynamical models when a small noise perturbs the convectively unstable uniform flow.

In the cellular automata (CA) models, there should be an analogous phenomenon to the convective instability although the idea of linear stability does not exist; the effect of localized perturbation travels only backward in the frame moving with cars when car-car interaction usually determined by gap in front of a car [6,13]. On the other hand, the oscillatory flow is more difficult to realize because of the discreteness of the dynamical variables. Recently,

however, multi-value extension of CA models was considered and it was reported that the flow pattern similar to SGS can be realized by perturbing a meta-stable uniform state [6]. In such a CA model, the mechanism that causes the complex flow pattern discussed here may hold.

In the real traffic, it has been observed near an on-ramp that SF is followed by SGS towards the upper stream, i.e. each car experiences the fluctuating high flux flow after going through alternate jammed and free flow regions as approaching the on-ramp [2]. This can be interpreted as follows: First, the convectively unstable uniform flow region is formed near the on-ramp by the influx. Then, small noise in the flow induces the convectively stabilized oscillatory flow, which corresponds to SF. The oscillatory flow breaks up as it travels towards the upper stream, and many small jams are formed in the upper stream side of the oscillatory flow region, which is SGS. This pattern is maintained near the on-ramp by small noise, namely, the noise-sustained structure.

Summarizing our results, we have studied the convective instability of the uniform flow solution in the OV model and shown that a localized perturbation to it generates the sequence of flow patterns; the oscillatory flow followed by the alternating sequence of jams and free flows towards the upper stream. We have demonstrated that the oscillatory solution is linearly unstable but is stabilized convectively, and have clarified the selection mechanism of the oscillatory solution out of the possible range of wave length. It is shown that the real traffic flow pattern observed near an on-ramp can be interpreted using the idea of noise-sustained structure in the open flow system.

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- [14] The possibility of the pattern formation in a car-following model has been already pointed out in Ref. [9].
- [15] It is not clear that these two types of oscillatory flows have the same physical origin because, in the present model, only a few cars are involved within a period of the oscillation, therefore, the validity of the continuum description is not obvious.

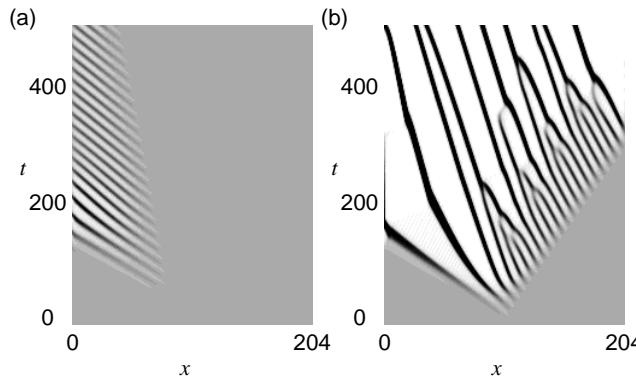


FIG. 1. The spatio-temporal diagrams of density. The horizontal axis is the position of a car x and the vertical axis is the time t . The higher density region is shown by a darker region. The darkness is adjusted so that the initial uniform flow region is shown by a gray region. (a) $a = 1.4$, $\bar{b} = 2.0$, $\epsilon = 0.1$, and $L = 204$. The disturbed region is convected only upstream. (b) $a = 1.0$, $\bar{b} = 2.0$, $\epsilon = 0.1$, and $L = 204$. The disturbed region spreads in both directions.

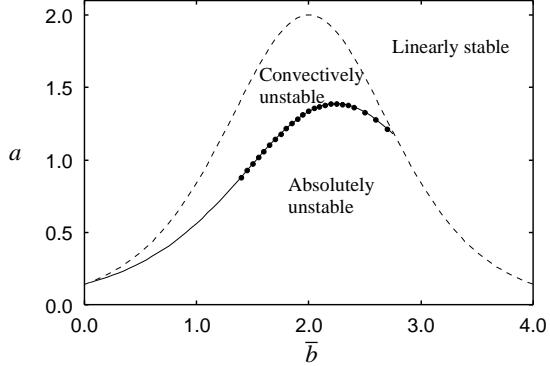


FIG. 2. The parameter region where the uniform solution is convectively unstable. The solid line is the parameter boundary $a = a_c(\bar{b})$ and the dashed line is the linear stability limit $a = 2U'(\bar{b})$. The uniform solution is convectively unstable in the region between the solid line and the dashed line. The open circles are the parameter $a = a_c(\bar{b})$ estimated by the numerical simulations.

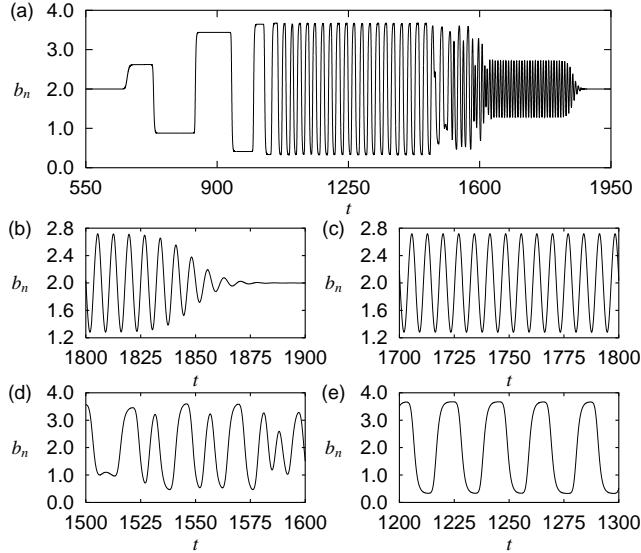


FIG. 3. The time evolution of the $n = -578$ th car's headway $b_n(t)$ with $a = 1$, $\bar{b} = 2$, $\epsilon = 0.1$, and $L = 800$. (a) The effect of the perturbation has not reached to the both ends of the system yet. Thus, the linearly unstable uniform flow regions are left and the whole structure can be seen. (b) The downstream front of the disturbed region. (c) The oscillatory flow region. (d) The oscillatory flow breaks up. (e) The alternating sequence of jams and free flows, which is not completely periodic.

TABLE I. The values of λ from Eq. (12) and from the results of the numerical simulation.

| \bar{b} | a | λ (Eq. (12)) | λ (simulation) |
|-----------|----------------------|----------------------|------------------------|
| 2.0 | 1.0 | 4.35 | 4.36 |
| 2.0 | 1.5 | 6.31 | 6.35 |
| 2.2 | $2U'(\bar{b}) - 1.0$ | 4.30 | 4.30 |
| 2.2 | $2U'(\bar{b}) - 0.5$ | 6.23 | 6.28 |
| 1.8 | $2U'(\bar{b}) - 0.5$ | 6.23 | 6.28 |